Complex numbers in standard form

Recall that the standard form of complex numbers is $a + bi$, where $a, b \in \mathbb{R}$

Exercise 1.1

Find the complex numbers in normal form corresponding to the following expressions:

a. $\left(\frac{1+i}{1-i}\right)^2$

b. $(1-i)(1+i)\frac{2}{2-i}$

c. $(-i)^{3253}$

d. $\frac{1-i^2+i^4-i^6+i^8-i^{10}}{1+i+i^2+i^3+i^4+i^5}$

e. $\sqrt{i}$

f. $\sqrt{-2i}$

g. $\sqrt{1+i\sqrt{3}}+\sqrt{1-i\sqrt{3}}$

Solution Exercise 1.1

a. $\left(\frac{1+i}{1-i}\right)^2 = \frac{2i}{-2i} = \frac{-2}{2} = -1$

b. $(1-i)(1+i)\frac{2}{2-i} = \frac{4}{2-i} = \frac{8+4i}{5} = \frac{8}{5}+\frac{4}{5}i$

c. because of the periodic nature of $(-i)^n = (-i)^{n+4}$ we can state $(-i)^n = (-i)^{n \bmod 4}$.

$(-i)^{3253 \bmod 4} = -i$

d. $\frac{1-i^2+i^4-i^6+i^8-i^{10}}{1+i+i^2+i^3+i^4+i^5} = \frac{6}{1+i} = 3-3i$

e. $\sqrt{i} = \left(e^{\frac{1}{2}\pi i + k2\pi}\right)^{\frac{1}{2}} = e^{\frac{1}{4}\pi i + k\pi} = \pm\left(\cos\left(\frac{1}{4}\pi\right) + i\sin\left(\frac{1}{4}\pi\right)\right) = \pm\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}i\right)$

f. $\sqrt{-2i} = i\sqrt{2}\sqrt{i} = i\sqrt{2}\left(\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}i\right) = -1+i$

g.
$$\begin{aligned}\sqrt{1+i\sqrt{3}}+\sqrt{1-i\sqrt{3}} &= \sqrt{2}e^{\frac{1}{6}\pi i}+\sqrt{2}e^{-\frac{1}{6}\pi i}\\ &= \sqrt{2}\left(\frac{1}{2}\sqrt{3}+\frac{1}{2}i\right)+\sqrt{2}\left(\frac{1}{2}\sqrt{3}-\frac{1}{2}i\right)\\ &= \sqrt{6}\end{aligned}$$